

Yetter-Drinfeld category for dual quasi-Hopf algebras

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Abstract. Let H be a dual quasi-Hopf algebra. In this paper we define all possible categories of Yetter-Drinfeld modules over H , and show that they are isomorphic. We prove also that the category ${}^H_H\mathcal{YD}^{fd}$ of finite-dimensional left-left Yetter-Drinfeld modules is rigid.

Keywords: Dual quasi-Hopf algebra; Yetter-Drinfeld module; Center construction; Rigid braid monoidal category.

Mathematics Subject Classification: 16W30.

Introduction

Quasi-bialgebras and quasi-Hopf algebras have been introduced by Drinfeld in [5], in connection with the Knizhnik-Zamolodchikov equations, and have been used afterwards in several branches of mathematics and physics. For the studies of dual quasi-Hopf algebras, one can refer to [3, 4, 6]. In a quasi-bialgebra H , the comultiplication is not coassociative but is quasi-coassociative in the sense that the comultiplication is associative up to conjugation by an invertible element in $H \otimes H \otimes H$. Hence the definition of a quasi-bialgebra and a quasi-Hopf algebra is not self-dual, inspired by which Majid in [7] introduced the notion of dual quasi-Hopf algebras.

For a dual quasi-Hopf algebra H , the category of right H -comodule \mathcal{M}^H is monoidal with the usual tensor product. The difference between a coquasi-Hopf algebra and a Hopf algebra lies in the fact that the associativity of tensor product in the category \mathcal{M}^H is not trivial, but modified by an invertible element $\sigma \in (H \otimes H \otimes H)^*$. Consequently the multiplication of H is no longer associative.

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In [8], the left Yetter-Drinfeld module over quasi-Hopf algebras was firstly constructed with the help of the isomorphism between the category of Yetter-Drinfeld modules and the center of the module category. Subsequently, the authors in [1] introduced all kinds of Yetter-Drinfeld modules and showed that the category of finite dimensional Yetter-Drinfeld modules is rigid.

Motivated by these ideas, in this paper, we will introduce the Yetter-Drinfeld modules over dual quasi-Hopf algebra H , Then we will prove that the category ${}^H_H\mathcal{YD}^{fd}$ of finite-dimensional left-left Yetter-Drinfeld modules is braided rigid.

This paper is organized as follows. In section 1, we will review some known results about dual quasi-Hopf algebras and monoidal category. In section 2, we will firstly introduce the notion of left-left Yetter-Drinfeld modules over dual quasi-Hopf algebra H . In particular, as in the case of Hopf algebras, H itself is made to be an object in ${}^H_H\mathcal{YD}$, the category of left-left Yetter-Drinfeld modules (see Proposition 2.6). For this purpose, we will prove some equations needed in our computations (see Lemma 2.1 and Lemma 2.2). Next some category isomorphisms will be established, including the isomorphisms between all kinds of categories of Yetter-Drinfeld modules and the centers of categories of H -comodules (see Proposition 2.7 and 2.10). Finally we verify that different categories of Yetter-Drinfeld modules are also isomorphic (see Proposition 2.12 and 2.13).

In section 3, we will verify that the category of finite dimensional left-left Yetter-Drinfeld modules is rigid, and give the explicit forms of the left and right duals of any object.

1 Preliminary

Throughout this article, let k be a fixed field. All algebras, coalgebras, linear spaces etc. will be over k ; unadorned \otimes means \otimes_k .

1.1 Dual quasi-Hopf algebra

Recall from [7] that a dual quasi-bialgebra H is a coassociative coalgebra with comultiplication Δ and counit ε together with coalgebra morphisms $m_H : H \otimes H \rightarrow H$ (the multiplication, we write $m_H(h \otimes h') = hh'$) and $\eta_H : k \rightarrow H$ (the unit, we write $\eta_H(1) = 1$), and a invertible element $\sigma \in (H \otimes H \otimes H)^*$ (the reassociator), such that for all $a, b, c, d \in H$ the following relations hold:

$$a_1(b_1c_1)\sigma(a_2, b_2, c_2) = \sigma(a_1, b_1, c_1)(a_2b_2)c_2, \quad (1.1)$$

$$1a = a1 = a, \quad (1.2)$$

$$\sigma(a_1, b_1, c_1d_1)\sigma(a_2b_2, c_2, d_2) = \sigma(b_1, c_1, d_1)\sigma(a_1, b_2c_2, d_2)\sigma(a_2, b_3, c_3), \quad (1.3)$$

$$\sigma(a, 1, b) = \varepsilon(a)\varepsilon(b). \quad (1.4)$$

H is called a dual quasi-Hopf algebra if, moreover, there exists an anti-morphism s of the coalgebra H and element $\alpha, \beta \in H^*$ such that for all $h \in H$,

$$s(h_1)\alpha(h_2)h_3 = \alpha(h)1, \quad h_1\beta(h_2)s(h_3) = \beta(h)1, \quad (1.5)$$

$$\sigma(h_1\beta(h_2), s(h_3), \alpha(h_4)h_5) = \sigma^{-1}(s(h_1), \alpha(h_2)h_3, \beta(h_4)s(h_5)) = \varepsilon(h). \quad (1.6)$$

It follows from the axioms that $s(1) = 1$ and $\alpha(1)\beta(1) = 1$. Moreover (1.3) and (1.4) imply that

$$\sigma(1, a, b) = \sigma(a, b, 1) = \varepsilon(a)\varepsilon(b). \quad (1.7)$$

Together with a dual quasi-Hopf algebra $H = (H, m, 1, \Delta, \varepsilon, \sigma, s, \alpha, \beta)$ with bijective antipode, we also have H^{op}, H^{cop} and $H^{op, cop}$ as dual quasi-Hopf algebras. The dual quasi-Hopf structures are obtained by putting $\sigma_{cop} = \sigma^{-1}, \sigma_{op} = (\sigma^{-1})^{321}$, and $\sigma_{op, cop} = \sigma^{321}$. $s_{op} = s_{cop} = (s_{op, cop}^{-1}) = s^{-1}$, $\alpha_{cop} = \beta s^{-1}$, $\alpha_{op} = \alpha s^{-1}$, $\alpha_{op, cop} = \beta$, $\beta_{cop} = \alpha s^{-1}$, $\beta_{op} = \beta s^{-1}$, $\beta_{op, cop} = \alpha$. Here $\sigma^{321}(a, b, c) = \sigma(c, b, a)$.

We recall that an invertible element $F \in (H \otimes H)^*$ satisfying $F(1, a) = F(a, 1) = \varepsilon(a)$, induces a twisted transformation

$$a \cdot b = F(a_1, b_1)a_2b_2F^{-1}(a_3, b_3), \quad (1.8)$$

$$\sigma_F(a, b, c) = F(b_1, c_1)F(a_1, b_2c_2)\sigma(a_2, b_3, c_3)F^{-1}(a_3b_4, c_4)F^{-1}(a_4, b_5) \quad (1.9)$$

For a Hopf algebra, one knows that the antipode is an anti-algebra morphism, i.e., $s(ab) = s(b)s(a)$. For a dual quasi-Hopf algebra, this is true only up to a twist, namely, there exists a twist transformation $f \in (H \otimes H)^*$ such that for all $a, b \in H$,

$$f(a_1, b_1)s(a_2b_2)g(a_3, b_3) = s(b)s(a). \quad (1.10)$$

The element f can be computed explicitly. For all $a, b, c, d \in H$, set

$$\nu(a, b, c, d) = \sigma(a_1, b_1, c_1)\sigma^{-1}(a_2b_2, c_2, d),$$

$$\mu(a, b, c, d) = \sigma(a_1b_1, c_1, d)\sigma^{-1}(a_2, b_2, c_2).$$

Define elements $\lambda, \chi \in (H \otimes H)^*$ by

$$\lambda(a, b) = \nu(s(b_1), s(a_1), a_3, b_3)\alpha(a_2)\alpha(b_2),$$

$$\chi(a, b) = \mu(a_1, b_1, s(b_3), s(a_3))\beta(a_2)\beta(b_2).$$

Then f and g are given by the following formulae:

$$f(a, b) = \sigma^{-1}(s(b_1)s(a_1), a_3b_3, s(a_5b_5))\lambda(a_2, b_2)\beta(a_4b_4),$$

$$g(a, b) = \sigma^{-1}(s(a_1b_1), a_3b_3, s(b_5)s(a_5))\chi(a_4, b_4)\alpha(a_2b_2).$$

The elements λ, χ and the twist f fulfill the relations

$$f(a_1, b_1)\alpha(a_2, b_2) = \lambda(a, b), \quad \beta(a_1, b_1)g(a_2, b_2) = \chi(a, b). \quad (1.11)$$

The corresponding reassociator is given by

$$\sigma_f(a, b, c) = \sigma(s(c), s(b), s(a)). \quad (1.12)$$

1.2 Monoidal categories and Center construction

A monoidal category means a category \mathcal{C} with objects U, V, W , etc., a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ equipped with an natural transformation consisting of functorial isomorphism $a_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$ satisfying a pentagon identity, and a compatible unit object I and associated functorial isomorphisms (the left and the right unit constraints, $l_V : V \cong V \otimes I$ and $r_V : V \cong I \otimes V$, respectively.) Now if \mathcal{C} and \mathcal{D} are monoidal categories then, roughly speaking, we say that $F : \mathcal{C} \rightarrow \mathcal{D}$ is a monoidal functor if it respects the tensor products (in the sense that for any two objects $U, V \in \mathcal{C}$ there exists a functorial isomorphism $\Psi : F(U) \otimes F(V) \rightarrow F(U \otimes V)$ such that Ψ respects the associativity constraints), the unit object and the left and right unit constraints (for a complete definition see [7]).

If H is a dual quasi-Hopf algebra, then the categories \mathcal{M}^H and ${}^H\mathcal{M}$ are monoidal categories. The associative constraint on \mathcal{M}^H is the following: for any $M, N, P \in \mathcal{M}^H$, and $m \in M, n \in N$, $a_{M,N,P} : (M \otimes N) \otimes P \rightarrow M \otimes (N \otimes P)$ is given by

$$a_{M,N,P}((m \otimes n) \otimes p) = \sigma(m_{(1)}, n_{(1)}, p_{(1)})m_{(0)} \otimes (n_{(0)} \otimes p_{(0)})$$

On ${}^H\mathcal{M}$, the associative constraint is given by

$$a_{M,N,P}((m \otimes n) \otimes p) = \sigma^{-1}(m_{(-1)}, n_{(-1)}, p_{(-1)})m_{(0)} \otimes (n_{(0)} \otimes p_{(0)}).$$

Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a monoidal category, and $V \in \mathcal{C}$. $V^* \in \mathcal{C}$ is called the left dual of V , if there exist two morphisms $ev_V : V^* \otimes V \rightarrow I$ and $coev_V : I \rightarrow V \otimes V^*$ such that

$$\begin{aligned} (V \otimes ev_V) \circ a_{V,V^*,V} \circ (coev_V \otimes V) &= V, \\ (ev_V \otimes V^*) \circ a_{V^*,V,V^*}^{-1} \circ (V^* \otimes coev_V) &= V^*. \end{aligned}$$

${}^*V \in \mathcal{C}$ is called a right dual of V if there exist two morphisms $ev'_V : V \otimes {}^*V \rightarrow I$ and $coev'_V : I \rightarrow {}^*V \otimes V$ such that

$$\begin{aligned} ({}^*V \otimes ev'_V) \circ a_{{}^*V,V,{}^*V} \circ (coev'_V \otimes {}^*V) &= {}^*V, \\ (ev'_V \otimes V) \circ a_{V,{}^*V,V}^{-1} \circ (V \otimes coev'_V) &= V. \end{aligned}$$

\mathcal{C} is called a rigid monoidal category if every object of \mathcal{C} has a left and right dual. The category ${}^H\mathcal{M}_{fd}$ of finite dimensional modules over a dual quasi-Hopf algebra H is rigid.

For $V \in {}^{\mathcal{H}}\mathcal{M}_{fd}$, $V^* = \text{Hom}(V, k)$ with left coaction $\lambda(\varphi) = \langle \varphi, v_{i(0)} \rangle s^{-1}(v_{i(-1)}) \otimes v^i$. The evaluation and coevaluation are given by

$$ev_V(\varphi \otimes v) = \beta(s^{-1}(v_{(-1)}))\varphi(v_{(0)}), \quad coev_V(1) = \alpha(s^{-1}(v_{i(-1)}))v_{i(0)} \otimes v^i. \quad (1.13)$$

where $\{v_i\}_i$ is a basis in V with dual basis $\{v^i\}_i$.

The right dual *V of V is the same dual vector space equipped with the left H -comodule structure given by $\lambda(\varphi) = \langle \varphi, v_{i(0)} \rangle s(v_{i(-1)}) \otimes v^i$ and

$$ev'_V(v \otimes \varphi) = \beta(v_{(-1)})\varphi(v_{(0)}), \quad coev'_V(1) = v^i \otimes \alpha(v_{i(-1)})v_{i(0)}. \quad (1.14)$$

For a braided monoidal category \mathcal{C} , let \mathcal{C}^{in} be equal to \mathcal{C} as a monoidal category, with the mirror-reversed braiding $\tilde{c}_{M,N} = c_{M,N}^{-1}$.

Following [8], the left weak center $\mathcal{W}_l(\mathcal{C})$ is the category with the objects $(V, s_{V,-})$, where $V \in \mathcal{C}$ and $s_{V,-} : V \otimes - \rightarrow - \otimes V$ is a family of natural transforms such that $s_{V,I} = id_V$ and for all $X, Y \in \mathcal{C}$

$$(X \otimes s_{V,Y}) \circ a_{X,V,Y} \circ (s_{V,X} \otimes Y) = a_{X,Y,V} \circ s_{V,X \otimes Y} \circ a_{V,X,Y}. \quad (1.15)$$

A morphism between $(V, s_{V,-})$ and $(V', s_{V',-})$ consists of $\psi : V \rightarrow V'$ in \mathcal{C} such that

$$(X \otimes \psi) \circ s_{V,X} = c_{V',X} \circ (\psi \otimes X).$$

$\mathcal{W}_l(\mathcal{C})$ is a prebraided monoidal category. The tensor product is

$$(V, s_{V,-}) \otimes (V', s_{V',-}) = (V \otimes V', s_{V \otimes V',-}),$$

with

$$s_{V \otimes V', X} = a_{X,V,V'} \circ (s_{V,X} \otimes V') \circ a_{V,X,V'}^{-1} \circ (V \otimes s_{V',X}) \circ a_{V,V',X}, \quad (1.16)$$

and the unit is (I, id) . The braiding s on $\mathcal{W}_l(\mathcal{C})$ is given by

$$c_{V,V'} = s_{V,V'} : (V, s_{V,-}) \otimes (V', s_{V',-}) \rightarrow (V', s_{V',-}) \otimes (V, s_{V,-}).$$

The center $\mathcal{Z}_l(\mathcal{C})$ is the full subcategory of $\mathcal{W}_l(\mathcal{C})$ consisting of objects $(V, s_{V,-})$ with $s_{V,-}$ a natural isomorphism. $\mathcal{Z}_l(\mathcal{C})$ is a braided monoidal category.

The right weak center $\mathcal{W}_r(\mathcal{C})$ is the category with the objects $(V, c_{-,V})$, where $V \in \mathcal{C}$ and $t_{-,V} : - \otimes V \rightarrow V \otimes -$ is a family of natural transforms such that $t_{I,V} = id_V$ and

$$a_{V,X,Y}^{-1} \circ t_{X \otimes Y,V} \circ a_{X,Y,V}^{-1} = (t_{X,V} \otimes Y) \circ a_{X,V,Y}^{-1} \circ (X \otimes t_{Y,V}), \quad (1.17)$$

for all $X, Y \in \mathcal{C}$. A morphism between $(V, t_{-,V})$ and $(V', t_{-,V'})$ consists of $\psi : V \rightarrow V'$ in \mathcal{C} such that

$$(\psi \otimes X) \circ t_{X,V} = t_{X,V'} \circ (X \otimes \psi).$$

$\mathcal{W}_r(\mathcal{C})$ is a prebraided monoidal category. The unit is (I, id) and the tensor product is

$$(V, t_{-,V}) \otimes (V', t_{-,V'}) = (V \otimes V', (V, t_{-,V \otimes V'}))$$

with

$$t_{-,V \otimes V'} = a_{V,V',X}^{-1} \circ (V \otimes t_{X,V'}) \circ a_{V,X,V'} \circ (t_{X,V} \otimes V') \circ a_{X,V,V'}^{-1}. \quad (1.18)$$

The braiding d is given by

$$d_{V,V'} = t_{V,V'} : (V, s_{-,V}) \otimes (V', s_{-,V'}) \rightarrow (V', s_{-,V'}) \otimes (V, s_{-,V}).$$

The center $\mathcal{Z}_r(\mathcal{C})$ is the full subcategory of $\mathcal{W}_r(\mathcal{C})$ consisting of objects $(V, t_{-,V})$ with $t_{-,V}$ a natural isomorphism. $\mathcal{Z}_r(\mathcal{C})$ is a braided monoidal category.

2 Yetter-Drinfeld modules over a dual quasi-Hopf algebra

In this section, we will construct all kinds of Yetter-Drinfeld modules over a dual-quasi Hopf algebra, and establish the isomorphism between the category of Yetter-Drinfeld modules and the center of comodule category.

In a Hopf algebra H , we obviously have the identity $h(g_1 s(g_2)) = h\varepsilon(g)$ for all $g, h \in H$. Now this formula could be generalized to the dual quasi-Hopf algebra setting as follows:

Lemma 2.1. *Let H be a dual quasi-Hopf algebra with bijective antipode s . Define elements p^R, q^R, p^L, q^L in $(H \otimes H)^*$ by*

$$\begin{aligned} p^R(a, b) &= \sigma^{-1}(a, b_1, s(b_3))\beta(b_2), & q^R(a, b) &= \sigma(a, b_3, s^{-1}(b_1))\alpha(s^{-1}(b_2)), \\ p^L(a, b) &= \sigma(s^{-1}(a_3), a_1, b)\beta(s^{-1}(a_2)), & q^L(a, b) &= \sigma^{-1}(s(a_1), a_3, b)\alpha(a_2), \end{aligned}$$

for all $a, b \in H$. Then they obey the following relations

$$p^R(a_1, b)a_2 = (a_1 b_1)p^R(a_2, b_2)s(a_3), \quad q^R(a_2, b)a_1 = (a_2 b_3)q^R(a_1, b_2)s^{-1}(b_1), \quad (2.1)$$

$$p^L(a, b_1)b_2 = s^{-1}(a_3)p^L(a_2, b_2)(a_1 b_1), \quad q^L(a, b_2)b_1 = s(a_1)q^L(a_2, b_1)(a_3 b_2), \quad (2.2)$$

and

$$q^R(a_1 b_1, s(b_3))p^R(a_2, b_2) = \varepsilon(a)\varepsilon(b), \quad p^L(s(a_1), a_3 b_2)q^L(a_2, b_1) = \varepsilon(a)\varepsilon(b), \quad (2.3)$$

$$q^L(s^{-1}(a_3), a_1 b_1)p^L(a_2, b_2) = \varepsilon(a)\varepsilon(b), \quad q^R(a_1, b_2)p^R(a_2 b_3, s^{-1}(b_1)) = \varepsilon(a)\varepsilon(b). \quad (2.4)$$

Moreover we have the following formulae

$$\begin{aligned} (1) \quad & q^R(a_1, b_1)q^R(a_2 b_2, c_1)\sigma^{-1}(a_3, b_3, c_3) \\ &= \sigma(a_2(b_4 c_4), s^{-1}(c_1), s^{-1}(b_1))f(s^{-1}(c_2), s^{-1}(b_2))q^R(a_1, b_3 c_3). \end{aligned} \quad (2.5)$$

$$\begin{aligned} (2) \quad & \sigma(a_1, b_1, c_1)p^R(a_2 b_2, c_2)p^R(a_3, b_3) \\ &= \sigma^{-1}(a_1(b_1 c_1), s(c_4), s(b_4))p^R(a_2, b_2 c_2)g(b_3, c_3). \end{aligned} \quad (2.6)$$

Proof. We only prove (2.5) and (2.6), since the proof of (2.1)–(2.4) is an easy exercise.

Define a map $\omega : (H^{\otimes 5})^* \rightarrow (H^{\otimes 3})^*$ by

$$\omega(\varphi)(a, b, c) = \varphi(a, b_3, c_3, s^{-1}(c_1), s^{-1}(b_1))\alpha(s^{-1}(b_2))\alpha(s^{-1}(c_2)),$$

for all $\varphi \in (H^{\otimes 5})^*$ and $a, b, c \in H$.

The left side of (2.5) is equal to $\omega(X)$, where for all $a, b, c, d, e \in H$,

$$X(a, b, c, d, e) = \sigma(a_1, b_1, e)\sigma(a_2b_2, c_1, d)\sigma^{-1}(a_3, b_3, c_2).$$

Since

$$\begin{aligned} & f(s^{-1}(c_1), s^{-1}(b_1))q^R(a, b_2c_2) \\ &= f(s^{-1}(c_1), s^{-1}(b_1))\sigma(a, b_4c_4, s^{-1}(b_2c_2))\alpha(s^{-1}(b_3c_3)) \\ &\stackrel{(1.10)}{=} \sigma(a, b_4c_4, s^{-1}(c_1)s^{-1}(b_1))f(s^{-1}(c_3), s^{-1}(b_3))\alpha(s^{-1}(c_2)s^{-1}(b_2)) \\ &\stackrel{(1.11)}{=} \sigma(a, b_3c_3, s^{-1}(c_1)s^{-1}(b_1))\lambda(s^{-1}(c_2), s^{-1}(b_2)) \\ &= \sigma(a, b_5c_5, s^{-1}(c_1)s^{-1}(b_1))\nu(b_4, c_4, s^{-1}(b_2), s^{-1}(b_2))\alpha(s^{-1}(b_3))\alpha(s^{-1}(c_3)), \end{aligned}$$

we have the right side of (2.5):

$$\begin{aligned} & \sigma(a_2(b_4c_4), s^{-1}(c_1), s^{-1}(b_1))f(s^{-1}(c_2), s^{-1}(b_2))q^R(a_1, b_3c_3) \\ &= \sigma(a_2(b_8c_9), s^{-1}(c_1), s^{-1}(b_1))\sigma(a_1, b_7c_8, s^{-1}(c_2)s^{-1}(b_2)) \\ & \quad \sigma(b_5, c_6, s^{-1}(c_4))\sigma^{-1}(b_6c_7, s^{-1}(c_3), s^{-1}(b_3))\alpha(s^{-1}(b_4))\alpha(s^{-1}(c_5)). \end{aligned}$$

Define an element $Y \in (H^{\otimes 5})^*$ by

$$Y(a, b, c, d, e) = \sigma^{-1}(c_1, d_1, e_1)\sigma(b_1, c_2, d_2e_2)\sigma(a_1, b_2c_3, d_3e_3)\sigma(a_2(b_3c_4), d_4, e_4).$$

Then

$$\begin{aligned} \omega(Y)(a, b, c) &= Y(a, b_3, c_3, s^{-1}(c_1), s^{-1}(b_1))\alpha(s^{-1}(b_2))\alpha(s^{-1}(c_2)) \\ &= \sigma^{-1}(c_6, s^{-1}(c_4), s^{-1}(b_4))\sigma(b_6, c_7, s^{-1}(c_3)s^{-1}(b_3))\sigma(a_1, b_7c_8, s^{-1}(c_2)s^{-1}(b_2)) \\ & \quad \sigma(a_2(b_8c_9), s^{-1}(c_1), s^{-1}(b_1))\alpha(s^{-1}(b_5))\alpha(s^{-1}(c_5)) \\ &\stackrel{(1.3)(1.5)}{=} \sigma(a_2(b_8c_9), s^{-1}(c_1), s^{-1}(b_1))\sigma(a_1, b_7c_8, s^{-1}(c_2)s^{-1}(b_2)) \\ & \quad \sigma(b_5, c_6, s^{-1}(c_4))\sigma^{-1}(b_6c_7, s^{-1}(c_3), s^{-1}(b_3))\alpha(s^{-1}(b_4))\alpha(s^{-1}(c_5)), \end{aligned}$$

which is equal to the right side of (2.5). Moreover

$$\begin{aligned} & Y(a, b, c, d, e) \\ &\stackrel{(1.3)}{=} \sigma^{-1}(c_1, d_1, e_1)\sigma(a_1, b_1, c_2(d_2e_2))\sigma(a_2b_2, c_3, d_3e_3)\sigma^{-1}(a_3, b_3, c_4)\sigma(a_4(b_4c_5), d_4, e_4) \end{aligned}$$

$$\stackrel{(1.1)}{=} \sigma(a_1, b_1, (c_1 d_1) e_1) \sigma^{-1}(c_2, d_2, e_2) \sigma(a_2 b_2, c_3, d_3 e_3) \sigma((a_3 b_3) c_4, d_4, e_4) \sigma^{-1}(a_4, b_4, c_5)$$

$$\stackrel{(1.3)}{=} \sigma(a_1, b_1, (c_1 d_1) e_1) \sigma(a_2 b_2, c_2 d_2, e_2) \sigma(a_3 b_3, c_3, d_3) \sigma^{-1}(a_4, b_4, c_4).$$

Then

$$\begin{aligned} \omega(Y)(a, b, c) &= \sigma(a_1, b_4, (c_5 \alpha(s^{-1}(c_4)) s^{-1}(c_3)) s^{-1}(b_2)) \sigma(a_2 b_5, c_6 s^{-1}(c_2), s^{-1}(b_1)) \\ &\quad \sigma(a_3 b_6, c_7, s^{-1}(c_1)) \sigma^{-1}(a_4, b_7, c_8) \alpha(s^{-1}(b_3)) \\ &\stackrel{(1.5)}{=} \sigma(a_1, b_3, s^{-1}(b_1)) \sigma(a_2 b_4, c_3, s^{-1}(c_1)) \sigma^{-1}(a_3, b_5, c_4) \alpha(s^{-1}(b_3)) \alpha(s^{-1}(c_2)) \\ &\stackrel{(1.3)(1.5)}{=} \sigma(a_1, b_3, s^{-1}(b_1)) \sigma(b_4, c_4, s^{-1}(c_2)) \sigma(a_2, b_5 c_5, s^{-1}(c_1)) \alpha(s^{-1}(b_2)) \alpha(s^{-1}(c_3)) \\ &\stackrel{(1.5)}{=} \sigma(a_1, b_3, s^{-1}(b_1)) \sigma^{-1}(a_2, b_4, c_3 s^{-1}(c_3)) \sigma(b_5, c_6, s^{-1}(c_2)) \\ &\quad \sigma(a_3, b_6 c_7, s^{-1}(c_1)) \alpha(s^{-1}(b_2)) \alpha(s^{-1}(c_4)) \\ &\stackrel{(1.3)}{=} \sigma(a_1, b_3, s^{-1}(b_1)) \sigma(a_2 b_4, c_3, s^{-1}(c_1)) \sigma^{-1}(a_3, b_5, c_4) \alpha(s^{-1}(b_2)) \alpha(s^{-1}(c_2)) \\ &= \omega(X)(a, b, c). \end{aligned}$$

The relation (2.5) is obtained. Since H^{cop} is also a dual-quasi Hopf algebra, by (2.5) we have (2.6).

The proof is completed. □

Lemma 2.2. Define an element $U \in (H \otimes H)^*$ by

$$U(a, b) = g(a_1, b_1) q^R(s(b_2), s(a_2)),$$

for all $a, b \in H$. Then we have the relation

$$U(a, b_1) s(b_2) = s(a_1 b_1) U(a_2, b_2) a_3, \quad (2.7)$$

$$U(a, b_1 c_1) U(b_2, c_2) = \sigma(a_1, b_1, c_1) \sigma(s((a_2 b_2) c_2), a_4, c_4) U(a_3 b_3, c_3). \quad (2.8)$$

Proof. The verification of (2.7) is straightforward. We only prove (2.8). For all $a, b, c \in H$,

$$\begin{aligned} &\sigma^{-1}(a_1, b_1, c_1) U(a_2, b_2 c_2) U(b_3, c_3) \\ &= \sigma^{-1}(a_1, b_1, c_1) g(a_2, b_2 c_2) q^R(s(b_3 c_3), s(a_3))) g(b_4, c_4) q^R(s(c_5), s(b_5)) \\ &\stackrel{(1.10)}{=} \sigma^{-1}(a_1, b_1, c_1) g(a_2, b_2 c_2) g(b_3, c_3) q^R(s(c_4) s(b_4), s(a_3))) q^R(s(c_5), s(b_5)) \\ &\stackrel{(1.12)}{=} g(a_1 b_1, c_1) g(b_2, c_2) \sigma^{-1}(s(c_2), s(b_3), s(a_3)) q^R(s(c_3) s(b_4), s(a_4))) q^R(s(c_4), s(b_5)) \\ &\stackrel{(2.5)}{=} g(a_1 b_1, c_1) g(b_2, c_2) \sigma(s(c_2)(s(b_3), s(a_3)), a_6, b_6) f(a_5, b_5) q^R(s(c_3), s(b_4) s(a_4)) \\ &= \sigma(s((a_1 b_1) c_1), a_6, b_6) g(a_2 b_2, c_2) g(b_3, c_3) f(a_4, b_4) q^R(s(c_3), s(a_3 b_3)) \\ &= \sigma(s((a_1 b_1) c_1), a_6, b_6) g(a_2 b_2, c_2) q^R(s(c_3), s(a_3 b_3)). \end{aligned}$$

The proof is completed. □

Definition 2.3. Let H be a dual quasi-Hopf algebra. A k -space M is called a left-left Yetter-Drinfeld module if M is a left H -comodule (denote the left coaction by $\lambda_M : M \rightarrow H \otimes M$, $m \mapsto m_{(-1)} \otimes m_{(0)}$) and H acts on M from the left (denote the left action by $h \cdot m$) such that for all $m \in M$ and $h \in H$ the following conditions hold:

$$\begin{aligned} & \sigma(h_1, g_1, m_{(-1)})\sigma((h_2 g_2 \cdot m_{(0)})_{(-1)}, h_3, g_3)(h_2 g_2 \cdot m_{(0)})_{(0)} \\ &= \sigma(h_1, (g_1 \cdot m)_{(-1)}, g_2)h_2 \cdot (g_1 \cdot m)_{(0)}, \end{aligned} \quad (2.9)$$

$$1_H \cdot m = m, \quad (2.10)$$

$$h_1 m_{(-1)} \otimes h_2 \cdot m_{(0)} = (h_1 \cdot m)_{(-1)} h_2 \otimes (h_1 \cdot m)_{(0)}, \quad (2.11)$$

for all $h, g \in H$ and $m \in M$. The category of left-left Yetter-Drinfeld modules over H is denoted by ${}^H_H\mathcal{YD}$.

Remark 2.4. In the above definition, if σ is trivial, we can recover the left-left Yetter-Drinfeld modules for Hopf algebras.

Proposition 2.5. Let H be a dual quasi-Hopf algebra, $M \in {}^H\mathcal{M}$, and $\cdot : H \otimes M \rightarrow M$ a k -linear map satisfying (2.9)–(2.10). Then (2.11) is equivalent to

$$\begin{aligned} (h \cdot m)_{(-1)} \otimes (h \cdot m)_{(0)} &= q^R((h_1 m_{(-1)})_1, s(h_5))(h_1 m_{(-1)})_2 s(h_4) \\ &\quad \otimes p^R((h_2 \cdot m_{(0)})_{(-1)}, h_3)(h_2 \cdot m_{(0)})_{(0)}, \end{aligned} \quad (2.12)$$

for all $h \in H, m \in M$.

Proof. Suppose that (2.11) holds. Then for all $h \in H, m \in M$,

$$\begin{aligned} & q^R((h_1 m_{(-1)})_1, s(h_5))(h_1 m_{(-1)})_2 s(h_4) \otimes p^R((h_2 \cdot m_{(0)})_{(-1)}, h_3)(h_2 \cdot m_{(0)})_{(0)} \\ & \stackrel{(2.11)}{=} q^R((h_1 \cdot m)_{(-1)1} h_2, s(h_6))[(h_1 \cdot m)_{(-1)2} h_3] s(h_5) \otimes p^R((h_1 \cdot m)_{(0)(-1)}, h_4)(h_1 \cdot m)_{(0)(0)} \\ &= q^R((h_1 \cdot m)_{(-1)1} h_2, s(h_6))[(h_1 \cdot m)_{(-1)2} h_3] s(h_5) \otimes p^R((h_1 \cdot m)_{(-1)3}, h_4)(h_1 \cdot m)_{(0)} \\ & \stackrel{(2.1)}{=} q^R((h_1 \cdot m)_{(-1)1} h_2, s(h_4))p^R((h_1 \cdot m)_{(-1)2}, h_3)(h_1 \cdot m)_{(-1)3} \otimes (h_1 \cdot m)_{(0)} \\ & \stackrel{(2.3)}{=} (h \cdot m)_{(-1)} \otimes (h \cdot m)_{(0)}. \end{aligned}$$

Conversely, assume that (2.12) holds. Particularly we have

$$h \cdot m = q^R(h_1 m_{(-1)}, s(h_4))p^R((h_2 \cdot m_{(0)})_{(-1)}, h_3)(h_2 \cdot m_{(0)})_{(0)}. \quad (2.13)$$

Thus for all $h \in H, m \in M$,

$$\begin{aligned} & (h_1 \cdot m)_{(-1)} h_2 \otimes (h_1 \cdot m)_{(0)} \\ & \stackrel{(2.12)}{=} q^R((h_1 m_{(-1)})_1, s(h_5))[(h_1 m_{(-1)})_2 s(h_4)] h_6 \otimes p^R((h_2 \cdot m_{(0)})_{(-1)}, h_3)(h_2 \cdot m_{(0)})_{(0)} \end{aligned}$$

$$\begin{aligned}
& \stackrel{(2.1)}{=} q^R((h_1 m_{(-1)})_2, s(h_4))(h_1 m_{(-1)})_1 \otimes p^R((h_2 \cdot m_{(0)})_{(-1)}, h_3)(h_2 \cdot m_{(0)})_{(0)} \\
& = q^R(h_2 m_{(0)(-1)}, s(h_5))h_1 m_{(-1)} \otimes p^R((h_3 \cdot m_{(0)(0)})_{(-1)}, h_4)(h_3 \cdot m_{(0)(0)})_{(0)} \\
& \stackrel{(2.13)}{=} h_1 m_{(-1)} \otimes h_2 \cdot m_{(0)}.
\end{aligned}$$

The proof is completed. \square

Proposition 2.6. *Let H be a dual quasi-Hopf algebra. Then H is an object in the category ${}^H_H\mathcal{YD}$ with the structures:*

$$\lambda(h) = h_1 s(h_3) \otimes h_2, \quad (2.14)$$

$$h \triangleright h' = \sigma(h_1, h'_1, s(h'_7))\sigma(h_2 h'_2, s(h'_5)s(h_5), h_7)g(h_4, h'_4)q^R(s(h'_6), s(h_6))h_3 h'_3, \quad (2.15)$$

for all $h, h' \in H$.

Proof. It is easy to see that H is a left H -comodule via λ . By the element U defined in Lemma 2.2, we obtain that

$$h \triangleright h' = \sigma(h_1, h'_1, s(h'_6))\sigma(h_2 h'_2, s(h_4 h'_4), h_6)U(h_5, h'_5)h_3 h'_3. \quad (2.16)$$

For all $a, b, h \in H$, we have

$$\begin{aligned}
& \sigma(a_1, b_1, h_1 s(h_3))\sigma((a_2 b_2 \triangleright h_2)_{(-1)}, a_3, b_3)(a_2 b_2 \triangleright h_2)_{(0)} \\
& = \sigma(a_1, b_1, h_1 s(h_8))\sigma(((a_4 b_4)h_4)_{(-1)}, a_8, b_8)\sigma(a_2 b_2, h_2, s(h_7)) \\
& \quad \sigma((a_3 b_3)h_3, s((a_5 b_5)h_5), a_7 b_7)U(a_6 b_6, h_6)((a_4 b_4)h_4)_{(0)} \\
& = \sigma(a_1, b_1, h_1 s(h_{10}))\sigma(a_2 b_2, h_2, s(h_9)) \\
& \quad \sigma((a_3 b_3)h_3, s((a_7 b_7)h_7), a_9 b_9)\sigma(((a_4 b_4)h_4)s((a_6 b_6)h_6), a_{10}, b_{10})U(a_8 b_8, h_8)(a_5 b_5)h_5 \\
& \stackrel{(1.3)}{=} \sigma(a_1, b_1, h_1 s(h_{11}))\sigma(a_2 b_2, h_2, s(h_{10}))\sigma(s((a_8 b_8)h_8), a_{10}, b_{10}) \\
& \quad \sigma((a_3 b_3)h_3, s((a_7 b_7)h_7)a_{11}, b_{11})\sigma((a_4 b_4)h_4, s((a_6 b_6)h_6), a_{12})U(a_9 b_9, h_9)(a_5 b_5)h_5 \\
& \stackrel{(1.1)}{=} \sigma(a_1, b_1, h_1 s(h_{13}))\sigma(a_2 b_2, h_2, s(h_{12}))\sigma(a_9, b_9, h_9)\sigma(s((a_{10} b_{10})h_{10}), a_{12}, b_{12})U(a_{11} b_{11}, h_{11}) \\
& \quad \sigma^{-1}(a_3, b_3, h_3)\sigma(a_4(b_4 h_4), s(a_8(b_8 h_8))a_{13}, b_{13})\sigma(a_5(b_5 h_5), s(a_7(b_7 h_7)), a_{14})a_6(b_6 h_6) \\
& \stackrel{(2.8)}{=} \sigma(a_1, b_1, h_1 s(h_{12}))\sigma(a_2 b_2, h_2, s(h_{11}))U(a_9, b_9 h_9)U(b_{10}, h_{10}) \\
& \quad \sigma^{-1}(a_3, b_3, h_3)\sigma(a_4(b_4 h_4), s(a_8(b_8 h_8))a_{10}, b_{11})\sigma(a_5(b_5 h_5), s(a_7(b_7 h_7)), a_{11})a_6(b_6 h_6) \\
& \stackrel{(1.3)}{=} \sigma(b_1, h_1, s(h_{12}))\sigma(a_1, b_2 h_2, s(h_{11}))U(a_7, b_8 h_9)U(b_9, h_{10}) \\
& \quad \sigma(a_2(b_3 h_4), s(a_6(b_7 h_8))a_8, b_{10})\sigma(a_3(b_4 h_5), s(a_5(b_6 h_7)), a_9)a_4(b_5 h_6) \\
& \stackrel{(2.7)}{=} \sigma(b_1, h_1, s(h_{11}))\sigma(a_1, b_2 h_2, s(h_{10}))U(a_6, b_7 h_7)U(b_9, h_9)
\end{aligned}$$

$$\begin{aligned}
& \sigma(a_2(b_3h_3), s(b_8h_8), b_{10})\sigma(a_3(b_4h_4), s(a_5(b_6h_6)), a_7)a_4(b_5h_5) \\
& \stackrel{(2.7)}{=} \sigma(b_1, h_1, s(h_{11}))\sigma(a_1, b_2h_2, s(h_{10}))\sigma(a_2(b_3h_3), s(b_8h_8), b_{10})U(b_9, h_9) \\
& \sigma(a_3(b_4h_4), s(a_5(b_6h_6)), a_7)U(a_6, b_7h_7)a_4(b_5h_5) \\
& = \sigma(b_1, h_1, s(h_{11}))\sigma(a_1, b_2h_2, s(b_9h_9)b_{11})\sigma(a_2(b_3h_3), s(b_8h_8), b_{12})U(b_{10}, h_{10}) \\
& \sigma(a_3(b_4h_4), s(a_5(b_6h_6)), a_7)U(a_6, b_7h_7)a_4(b_5h_5) \\
& \stackrel{(1.3)}{=} \sigma(b_2h_2, s(b_{11}h_{11}), b_{13})\sigma(a_1, (b_3h_3)s(b_{10}h_{10}), b_{14})\sigma(a_2, b_4h_4, s(b_9h_9))U(b_{12}, h_{12}) \\
& \sigma(b_1, h_1, s(h_{13}))\sigma(a_3(b_5h_5), s(a_5(b_7h_7)), a_7)U(a_6, b_8h_8)a_4(b_6h_6) \\
& = \sigma(a_1, (b_1 \triangleright h)_{(-1)}, b_2)a_2 \triangleright (b_1 \triangleright h)_{(0)}.
\end{aligned}$$

Thus the relation (2.3) is satisfied. And

$$\begin{aligned}
& a_1h_{(-1)} \otimes a_2 \triangleright h_{(0)} = a_1(h_1s(h_3)) \otimes a_2 \triangleright h_2 \\
& = a_1(h_1s(h_8))\sigma(a_2, h_2, s(h_7))\sigma(a_3h_3, s(a_5h_5), a_7)U(a_6, h_6) \otimes a_4h_4 \\
& \stackrel{(1.1)}{=} \sigma(a_1, h_1, s(h_8))(a_2h_2)s(h_7)\sigma(a_3h_3, s(a_5h_5), a_7)U(a_6, h_6) \otimes a_4h_4 \\
& \stackrel{(2.7)}{=} \sigma(a_1, h_1, s(h_8))\sigma(a_3h_3, s(a_5h_5), a_9)(a_2h_2)[s(a_6h_6)a_8]U(a_7, h_7) \otimes a_4h_4 \\
& \stackrel{(1.1)}{=} \sigma(a_1, h_1, s(h_8))\sigma(a_2h_2, s(a_6h_6), a_8)[(a_3h_3)s(a_5h_5)]U(a_7, h_7)a_9 \otimes a_4h_4 \\
& = (a_1 \triangleright h)_{(-1)}a_2 \otimes (a_1 \triangleright h)_{(0)}.
\end{aligned}$$

The relation (2.9) is satisfied. Obviously $1_H \triangleright h = h$. Hence H is an object in the category ${}^H_H\mathcal{YD}$. The proof is completed. \square

Proposition 2.7. *Let H be a dual quasi-bialgebra and ${}^H\mathcal{M}$ the category of left H -comodules. Then we have category isomorphism ${}^H_H\mathcal{YD} \cong \mathcal{W}_r({}^H\mathcal{M})$.*

Proof. For any object $(M, c_{-,M})$ in the center $\mathcal{W}_r({}^H\mathcal{M})$, we adopt the notation

$$c_{N,M}(n \otimes m) = m_c \otimes n_c.$$

Define the left action of H on M by

$$h \cdot m = (id \otimes \varepsilon)c_{H,M}(h \otimes m) = \varepsilon(h_c)m_c.$$

for all $h \in H$. For any $\varphi \in N^* = \text{Hom}(N, k)$, there corresponds a linear map $\tilde{\varphi} : N \rightarrow H$ given by

$$\tilde{\varphi}(n) = \varphi(n_{(0)})n_{(-1)}.$$

Easy to see that $\tilde{\varphi}$ is left H -colinear. Hence by the naturality of $c_{-,M}$, we have

$$(id \otimes \tilde{\varphi}) \circ c_{N,M} = c_{H,M} \circ (\tilde{\varphi} \otimes id),$$

that is, $m_c \otimes \varphi(n_{c(0)})n_{c(-1)} = \varphi(n_{(0)})c_{H,M}(n_{(-1)} \otimes m)$. Applying $id \otimes \varepsilon$ to this equation, we obtain

$$m_c \otimes \varphi(n_c) = \varphi(n_{(0)})n_{(-1)} \cdot m.$$

By the arbitrariness of φ , we have

$$c_{N,M}(n \otimes m) = n_{(-1)} \cdot m \otimes n_{(0)}. \quad (2.17)$$

Since $\eta : k \rightarrow H$ is a morphism of coalgebra, by the naturality of $c_{-,M}$ and $c_{k,M} = id_M$, we have $1_H \cdot m = m$. By (1.17), we have

$$a_{M,H,H}^{-1} \circ c_{H \otimes H, M} \circ a_{H,H,M}^{-1} = (c_{H,M} \otimes H) \circ a_{H,M,H}^{-1} \circ (H \otimes c_{H,M}),$$

which evaluating at any element $(h \otimes g \otimes m)$ is (2.9). Finally because $c_{H,M}$ is a left H -coalgebra map, we have

$$c_{H,M}(h \otimes m)_{(-1)} \otimes c_{H,M}(h \otimes m)_{(0)} = h_1 m_{(-1)} \otimes c_{H,M}(h_2 \otimes m_{(0)}),$$

thus

$$(h_1 \cdot m)_{(-1)} h_2 \otimes (h_1 \cdot m)_{(0)} \otimes h_3 = h_1 m_{(-1)} \otimes h_2 m_{(0)} \otimes h_3.$$

Applying $id \otimes id \otimes \varepsilon$ to the above equation, we get (2.11). Hence M is a left-left Yetter-Drinfeld module. Therefore we have the functor

$$F : \mathcal{W}_r(\mathcal{H}\mathcal{M}) \longrightarrow {}^H_H\mathcal{YD}, \quad F((M, c_{-,M})) = M.$$

Conversely, assume that M is a left-left Yetter-Drinfeld module. For any left H -comodule N , give the linear map $c_{N,M} : N \otimes M \rightarrow M \otimes N$ by (2.17). Then it is straightforward to verify that $(M, c_{-,M})$ is an object of $\mathcal{W}_r(\mathcal{H}\mathcal{M})$. Therefore we have the functor

$$G : {}^H_H\mathcal{YD} \longrightarrow \mathcal{W}_r(\mathcal{H}\mathcal{M}), \quad G(M) = (M, c_{-,M}).$$

Obviously F and G induce a category isomorphism. \square

The prebraided monoidal structure on $\mathcal{W}_r(\mathcal{H}\mathcal{M})$ induces a monoidal structure on ${}^H_H\mathcal{YD}$ such that the forgetful functor ${}^H_H\mathcal{YD} \rightarrow {}^H M$ is monoidal. Using (1.18), we find that the action on the tensor product $M \otimes N$ of two left-left Yetter-Drinfeld modules M and N is given by

$$\begin{aligned} h \cdot (m \otimes n) = & \sigma(h_1, m_{(-1)}, n_{(-1)1}) \sigma^{-1}((h_2 \cdot m_{(0)})_{(-1)1}, h_3, n_{(-1)2}) \\ & \sigma((h_2 \cdot m_{(0)})_{(-1)2}, (h_4 \cdot n_{(0)})_{(-1)}, h_5) (h_2 \cdot m_{(0)})_{(0)} \otimes (h_4 \cdot n_{(0)})_{(0)}, \end{aligned} \quad (2.18)$$

for all $m \in M, n \in N$. The braiding is given by

$$c_{N,M}(n \otimes m) = n_{(-1)} \cdot m \otimes n_{(0)}. \quad (2.19)$$

Furthermore we have the following result.

Theorem 2.8. Suppose that H is a dual quasi-Hopf algebra, then $\mathcal{W}_r(\mathcal{H}\mathcal{M}) = \mathcal{Z}_l(\mathcal{H}\mathcal{M}) \cong {}^H_H\mathcal{YD}$, and the inverse $c_{N,M}^{-1} : M \otimes N \rightarrow N \otimes M$ of the braiding is given by

$$\begin{aligned} c_{N,M}^{-1}(m \otimes n) = & q^L(s^{-1}(n_{(-1)6}), m_{(-1)1}n_{(-1)1})\sigma(s^{-1}(n_{(-1)5}), m_{(-1)2}, n_{(-1)2}) \\ & p^R((s^{-1}(n_{(-1)4}) \cdot m_{(0)})_{(-1)}, s^{-1}(n_{(-1)3}))n_{(0)} \otimes (s^{-1}(n_{(-1)4}) \cdot m_{(0)})_{(0)}. \end{aligned} \quad (2.20)$$

Proof. For all $m \in M, n \in N$,

$$\begin{aligned} c_{N,M}^{-1}(c_{N,M}(n \otimes m)) &= c_{N,M}^{-1}(n_{(-1)} \cdot m \otimes n_{(0)}) \\ &= q^L(s^{-1}(n_{(-1)7}), (n_{(-1)1} \cdot m)_{(-1)1}n_{(-1)2})\sigma(s^{-1}(n_{(-1)6}), (n_{(-1)1} \cdot m)_{(-1)2}, n_{(-1)3}) \\ &\quad p^R((s^{-1}(n_{(-1)5}) \cdot (n_{(-1)1} \cdot m)_{(0)})_{(-1)}, s^{-1}(n_{(-1)4}))n_{(0)} \otimes (s^{-1}(n_{(-1)5}) \cdot (n_{(-1)1} \cdot m)_{(0)})_{(0)} \\ &= q^L(s^{-1}(n_{(-1)7}), (n_{(-1)1} \cdot m)_{(-1)}n_{(-1)2})\sigma(s^{-1}(n_{(-1)6}), (n_{(-1)1} \cdot m)_{(0)(-1)}, n_{(-1)3}) \\ &\quad p^R((s^{-1}(n_{(-1)5}) \cdot (n_{(-1)1} \cdot m)_{(0)(0)})_{(-1)}, s^{-1}(n_{(-1)4}))n_{(0)} \otimes (s^{-1}(n_{(-1)5}) \cdot (n_{(-1)1} \cdot m)_{(0)(0)})_{(0)} \\ &\stackrel{(2.11)}{=} q^L(s^{-1}(n_{(-1)7}), n_{(-1)1}m_{(-1)})\sigma(s^{-1}(n_{(-1)6}), (n_{(-1)2} \cdot m_{(0)})_{(-1)}, n_{(-1)3}) \\ &\quad p^R((s^{-1}(n_{(-1)5}) \cdot (n_{(-1)2} \cdot m_{(0)})_{(0)})_{(-1)}, s^{-1}(n_{(-1)4}))n_{(0)} \otimes (s^{-1}(n_{(-1)5}) \cdot (n_{(-1)2} \cdot m_{(0)})_{(0)})_{(0)} \\ &\stackrel{(2.9)}{=} q^L(s^{-1}(n_{(-1)9}), n_{(-1)1}m_{(-1)1}) \\ &\quad \sigma(s^{-1}(n_{(-1)8}), n_{(-1)2}, m_{(-1)2})\sigma((s^{-1}(n_{(-1)7})n_{(-1)3} \cdot m_{(0)})_{(-1)}, s^{-1}(n_{(-1)6}), n_{(-1)4}) \\ &\quad p^R((s^{-1}(n_{(-1)7})n_{(-1)3} \cdot m_{(0)})_{(0)(-1)}, s^{-1}(n_{(-1)5}))n_{(0)} \otimes (s^{-1}(n_{(-1)7})n_{(-1)3} \cdot m_{(0)})_{(0)(0)} \\ &= q^L(s^{-1}(n_{(-1)7}), n_{(-1)1}m_{(-1)1})\sigma(s^{-1}(n_{(-1)6}), n_{(-1)2}, m_{(-1)2})\beta(s^{-1}(n_{(-1)4})) \\ &\quad n_{(0)} \otimes s^{-1}(n_{(-1)5})n_{(-1)3} \cdot m_{(0)} \\ &\stackrel{(1.5)}{=} q^L(s^{-1}(n_{(-1)5}), n_{(-1)1}m_{(-1)1})\sigma(s^{-1}(n_{(-1)4}), n_{(-1)2}, m_{(-1)2})\beta(s^{-1}(n_{(-1)3})) \\ &\quad n_{(0)} \otimes m_{(0)} \\ &= q^L(s^{-1}(n_{(-1)3}), n_{(-1)1}m_{(-1)1})p^L(n_{(-1)2}, m_{(-1)2})n_{(0)} \otimes m_{(0)} \\ &\stackrel{(2.4)}{=} n \otimes m. \end{aligned}$$

That is, $c_{N,M}^{-1} \circ c_{N,M} = id_{N \otimes M}$. Similarly $c_{N,M} \circ c_{N,M}^{-1} = id_{M \otimes N}$.

The proof is completed. \square

We also introduce left-right, right-left and right-right Yetter-Drinfeld modules in the following definition.

Definition 2.9. Let H be a dual quasi-Hopf algebra.

- (1) A right-left Yetter-Drinfeld module is a left H -comodule M together with a right H -action \cdot on M such that for all $g, h \in H, m \in M$,

$$\sigma^{-1}(m_{(-1)}, h_1, g_1)\sigma^{-1}(h_3, g_3, (m \cdot h_2g_2)_{(-1)})(m \cdot h_2g_2)_{(0)}$$

$$= \sigma^{-1}(h_2, (m \cdot h_1)_{(-1)}, g_1)(m \cdot h_1)_{(0)} \cdot g_2, \quad (2.21)$$

$$m \cdot 1 = m, \quad (2.22)$$

$$h_2(m \cdot h_1)_{(-1)} \otimes (m \cdot h_1)_{(0)} = m_{(-1)}h_1 \otimes m_{(0)} \cdot h_2. \quad (2.23)$$

The category of right-left Yetter-Drinfeld modules over H is denoted by ${}^H\mathcal{YD}_H$.

- (2) A left-right Yetter-Drinfeld module is a right H -comodule M together with a left H -action \cdot on M such that for all $g, h \in H, m \in M$,

$$\begin{aligned} & \sigma^{-1}((h_2g_2 \cdot m_{(0)})_{(1)}, h_1, g_1)\sigma^{-1}(h_3, g_3, m_{(1)})(h_2g_2 \cdot m_{(0)})_{(0)} \\ &= \sigma^{-1}(h_2, (g_2 \cdot m)_{(1)}, g_1)h_1 \cdot (g_2 \cdot m)_{(0)}, \end{aligned} \quad (2.24)$$

$$1 \cdot m = m, \quad (2.25)$$

$$(h_2 \cdot m)_{(0)} \otimes (h_2 \cdot m)_{(1)}h_1 = h_1 \cdot m_{(0)} \otimes h_2m_{(1)}. \quad (2.26)$$

The category of left-right Yetter-Drinfeld modules over H is denoted by ${}_H\mathcal{YD}^H$.

- (3) A right-right Yetter-Drinfeld module is a right H -comodule M together with a right H -action \cdot on M such that for all $g, h \in H, m \in M$,

$$\begin{aligned} & \sigma(h_1, g_1, (m_{(0)} \cdot h_2g_2)_{(1)})\sigma(m_{(1)}, h_3, g_3)(m_{(0)} \cdot h_2g_2)_{(0)} \\ &= \sigma(h_1, (m \cdot h_2)_{(1)}, g_2)(m \cdot h_2)_{(0)} \cdot g_1, \end{aligned} \quad (2.27)$$

$$m \cdot 1 = m, \quad (2.28)$$

$$(m \cdot h_2)_{(0)} \otimes h_1(m \cdot h_2)_{(1)} = m_{(0)} \cdot h_1 \otimes m_{(1)}h_2. \quad (2.29)$$

The category of right-right Yetter-Drinfeld modules over H is denoted by \mathcal{YD}_H^H .

Similarly, for the above definitions we have the following result:

Theorem 2.10. *Let H be a dual quasi-bialgebra. Then we have the following category isomorphisms:*

$$\mathcal{W}_l({}^H\mathcal{M}) \cong {}^H\mathcal{YD}_H, \quad \mathcal{W}_r(\mathcal{M}^H) \cong {}_H\mathcal{YD}^H, \quad \mathcal{W}_l(\mathcal{M}^H) \cong \mathcal{YD}_H^H.$$

If H is a dual quasi-Hopf algebra, then these three weak centers are equal to the centers.

The prebraided monoidal structure on $\mathcal{W}_l({}^H\mathcal{M})$ induces a monoidal structure on ${}^H\mathcal{YD}_H$. Using (1.16), we find that the action on $M \otimes N$ of two right-left Yetter-Drinfeld modules M and N is given by

$$\begin{aligned} (m \otimes n) \cdot h &= \sigma^{-1}(m_{(-1)1}, n_{(-1)}, h_1)\sigma(m_{(-1)2}, h_3, (n_{(0)} \cdot h_2)_{(-1)1}) \\ & \quad \sigma^{-1}(h_5, (m_{(0)} \cdot h_4)_{(-1)}, (n_{(0)} \cdot h_2)_{(-1)2})(m_{(0)} \cdot h_4)_{(0)} \otimes (n_{(0)} \cdot h_2)_{(0)}, \end{aligned} \quad (2.30)$$

for all $h \in H, m \in M$, and $n \in N$. And

The braiding $d_{M,N} : M \otimes N \rightarrow N \otimes M$ is given by

$$d_{M,N}(m \otimes n) = n_{(0)} \otimes m \cdot n_{(-1)}. \quad (2.31)$$

In the case when H is a dual quasi-Hopf algebra, the inverse of $d_{M,N}$ is given by

$$\begin{aligned} d_{M,N}^{-1}(n \otimes m) &= q^R(n_{(-1)1}m_{(-1)1}, s(n_{(-1)6}))\sigma^{-1}(n_{(-1)2}, m_{(-1)2}, s(n_{(-1)5})) \\ &\quad p^L(s(n_{(-1)3}), (m_{(0)} \cdot s(n_{(-1)4}))_{(-1)})(m_{(0)} \cdot s(n_{(-1)4}))_{(0)} \otimes n_{(0)}. \end{aligned} \quad (2.32)$$

For the reason of completeness, let us also describe the prebraided monoidal structure on ${}^H\mathcal{YD}^H$ and \mathcal{YD}_H^H . For $M, N \in {}^H\mathcal{YD}^H$, the action on $M \otimes N$ is given by

$$\begin{aligned} h \cdot (m \otimes n) &= \sigma^{-1}(h_5, m_{(1)}, n_{(1)2})\sigma((h_4 \cdot m_{(0)})_{(1)2}, h_3, n_{(1)1}) \\ &\quad \sigma^{-1}((h_4 \cdot m_{(0)})_{(1)1}, (h_2 \cdot n_{(0)})_{(1)}, h_1)(h_4 \cdot m_{(0)})_{(0)} \otimes (h_2 \cdot n_{(0)})_{(0)}, \end{aligned}$$

and

$$(m \otimes n)_{(0)} \otimes (m \otimes n)_{(1)} = m_{(0)} \otimes n_{(0)} \otimes m_{(1)}n_{(1)},$$

the braiding is the following:

$$t_{M,N}(m \otimes n) = m_{(1)} \cdot n \otimes m_{(0)}.$$

Now take $M, N \in \mathcal{YD}_H^H$, the action on $M \otimes N$ is the following:

$$\begin{aligned} (m \otimes n) \cdot h &= \sigma(h_1, (m_{(0)} \cdot h_2)_{(1)}, (n_{(0)} \cdot h_4)_{(1)1})\sigma^{-1}(m_{(1)1}, h_3, (n_{(0)} \cdot h_4)_{(1)2}) \\ &\quad \sigma(m_{(1)2}, n_{(1)}, h_5)(m_{(0)} \cdot h_2)_{(0)} \otimes (n_{(0)} \cdot h_4)_{(0)}. \end{aligned}$$

The braiding is given by

$$\vartheta_{M,N}(m \otimes n) = n_{(0)} \otimes m \cdot n_{(1)}.$$

Proposition 2.11. [1, Proposition 1.1] *Let \mathcal{C} be a monoidal category. Then we have a braided isomorphism of braided monoidal categories $T : \mathcal{Z}_l(\mathcal{C}) \rightarrow \mathcal{Z}_r(\mathcal{C})^{in}$, given by*

$$T(V, s_{V,-}) = (V, s_{V,-}^{-1}), \quad \text{and} \quad T(v) = v.$$

Of course, the conclusion holds for the right center. By this isomorphism, we have the following result:

Proposition 2.12. *Let H be a dual quasi-Hopf algebra. Then we have an isomorphism of braided monoidal categories*

$$T : {}^H\mathcal{YD}_H^{in} \cong {}^H\mathcal{YD},$$

defined as follows. For $M \in {}^H\mathcal{YD}_H$, $T(M) = M$ as a left H -comodule; the left H -action is given by

$$h \triangleright m = q^R(h_1m_{(-1)1}, s(h_6))\sigma^{-1}(h_2, m_{(-1)2}, s(h_5))$$

$$p^L(s(h_3), (m_{(0)} \cdot s(h_4))_{(-1)})(m_{(0)} \cdot s(h_4))_{(0)},$$

for all $h \in H, m \in M$, where \cdot is the right action of H on M . The functor T sends a morphism to itself.

Proof. The functor T is just the composition of the isomorphisms

$${}^H\mathcal{YD}_H^{in} \rightarrow \mathcal{Z}_l(\mathcal{H}\mathcal{M})^{in} \rightarrow \mathcal{Z}_r(\mathcal{H}\mathcal{M}) \rightarrow {}^H\mathcal{YD}.$$

For $M \in {}^H\mathcal{YD}_H^{in}$, we compute the corresponding left-left Yetter-Drinfeld module structure on M is the following:

$$\begin{aligned} h \triangleright m &= (id \otimes \varepsilon)s_{M,H}^{-1}(h \otimes m) = (id \otimes \varepsilon)\tilde{c}_{M,H}(h \otimes m) = (id \otimes \varepsilon)d_{M,H}^{-1}(h \otimes m) \\ &= q^R(h_1 m_{(-1)1}, s(h_6))\sigma^{-1}(h_2, m_{(-1)2}, s(h_5))p^L(s(h_3), (m_{(0)} \cdot s(h_4))_{(-1)})(m_{(0)} \cdot s(h_4))_{(0)}, \end{aligned}$$

as claimed. \square

In the same way, we have the following result.

Proposition 2.13. *Let H be a dual quasi-Hopf algebra. Then the categories \mathcal{YD}_H^H and ${}_H\mathcal{YD}^{Hin}$ are isomorphic as braided monoidal categories.*

3 The rigid braided category ${}_H\mathcal{YD}^{fd}$

It is well known that the category of finite dimensional Yetter-Drinfeld modules over a Hopf algebra with bijective antipode is rigid. Since ${}^H\mathcal{M}_{fd}$ is rigid, the same result holds for the category of finite dimensional Yetter-Drinfeld modules over a dual quasi-Hopf algebra. In this section we will give the explicit forms.

Proposition 3.1. *[1, Proposition 1.3] Let \mathcal{C} be a rigid monoidal category. Then the weak left (respectively right) center of \mathcal{C} , and is a rigid braided monoidal category.*

For example, for any object $(V, c_{-,V}) \in \mathcal{Z}_r(\mathcal{C})$, ${}^*(V, c_{-,V}) = ({}^*V, c_{-,{}^*V})$, with $c_{-,{}^*V}$ given by the following composition:

$$\begin{aligned} c_{X,{}^*V} : X \otimes {}^*V &\xrightarrow{coev'_V \otimes (X \otimes {}^*V)} ({}^*V \otimes V) \otimes (X \otimes {}^*V) \\ &\xrightarrow{a_{{}^*V,V,(X \otimes {}^*V)}} {}^*V \otimes (V \otimes (X \otimes {}^*V)) \\ &\xrightarrow{{}^*V \otimes a_{V,X,{}^*V}^{-1}} {}^*V \otimes ((V \otimes X) \otimes {}^*V) \\ &\xrightarrow{{}^*V \otimes c_{V,X}^{-1} \otimes {}^*V} {}^*V \otimes ((X \otimes V) \otimes {}^*V) \end{aligned} \tag{3.1}$$

$$\begin{aligned}
& \xrightarrow{*V \otimes a_{V,X,*V}} *V \otimes (X \otimes (V \otimes *V)) \\
& \xrightarrow{a_{*V,V,(V \otimes *V)}^{-1}} (*V \otimes X) \otimes (V \otimes *V) \\
& \xrightarrow{*V \otimes V \otimes ev'_v} *V \otimes X.
\end{aligned}$$

Lemma 3.2. *Let H be a dual quasi-Hopf algebra. Then for all $a, b, c \in H$, the following relations hold:*

$$q^L(a_1, b_1 c_1) \sigma(a_2, b_2, c_2) = q^L(a_2, b_1) \sigma^{-1}(s(a_1), a_3 b_2, c), \quad (3.2)$$

$$p^R(s(a_1), a_3 b_3) q^L(a_2, b_2) q^L(b_1, s(a_4 b_4)) = f(a, b), \quad (3.3)$$

Proof. By the definition of q^L , it is easy to verify (3.2). Then for all $a, b \in H$,

$$\begin{aligned}
& p^R(s(a_1), a_3 b_3) q^L(a_2, b_2) q^L(b_1, s(a_4 b_4)) \\
& = \sigma^{-1}(s(a_1), a_3 b_3, s(a_5 b_5)) \beta(a_4 b_4) q^L(a_2, b_2) q^L(b_1, s(a_6 b_6)) \\
& \stackrel{(1.10, 1.11)}{=} \sigma^{-1}(s(a_1), a_3 b_3, s(b_5) s(a_5)) q^L(a_2, b_2) \chi(a_4, b_4) q^L(b_1, s(b_6) s(a_6)) f(a_7, b_7) \\
& = \sigma^{-1}(s(a_1), a_5 b_3, s(b_9) s(a_{10})) \sigma^{-1}(s(a_2), a_4, b_2) \alpha(a_3) \\
& \sigma(a_6 b_4, s(b_8), s(a_9)) \sigma^{-1}(a_7, b_5, s(b_7)) \beta(a_8) \beta(b_6) q^L(b_1, s(b_{10}) s(a_{11})) f(a_{12}, b_{11}) \\
& \stackrel{(1.3, 1.5)}{=} \sigma^{-1}(s(a_2), a_4, b_2) \sigma^{-1}(s(a_1), a_5 b_3, s(b_9) s(a_{10})) \sigma^{-1}(a_6, b_4, s(b_8) s(a_9)) \\
& \alpha(a_3) \sigma(b_5, s(b_7), s(a_8)) \beta(b_6) \beta(a_7) q^L(b_1, s(b_{10}) s(a_{11})) f(a_{12}, b_{11}) \\
& \stackrel{(1.3, 1.5)}{=} \sigma^{-1}(s(a_1), a_3, b_2(s(b_6) s(a_6))) \alpha(a_2) \sigma(b_3, s(b_5), s(a_5)) \\
& \beta(b_4) \beta(a_4) q^L(b_1, s(b_7) s(a_7)) f(a_8, b_8) \\
& \stackrel{(1.1, 1.5)}{=} \sigma^{-1}(s(a_1), a_3, s(a_5)) \alpha(a_2) \sigma(b_2, s(b_4), s(a_6)) \beta(b_3) \beta(a_4) q^L(b_1, s(b_5) s(a_7)) f(a_8, b_6) \\
& \stackrel{(1.1, 1.5)}{=} \sigma(b_2, s(b_4), s(a_1)) \beta(b_3) q^L(b_1, s(b_5) s(a_2)) f(a_3, b_6) \\
& = p^L(s(b_2), s(a_1)) q^L(b_1, s(b_3) s(a_2)) f(a_3, b_4) \\
& \stackrel{(2.4)}{=} f(a, b),
\end{aligned}$$

as needed. The proof is completed. \square

Theorem 3.3. *Let H be a dual quasi-Hopf algebra. Then ${}^H_H \mathcal{YD}(\mathcal{H})^{fd}$ is a braided monoidal rigid category. For a finite dimensional left-left Yetter-Drinfeld module M with basis $\{m_i\}_i$ and dual basis $\{m^i\}_i$, the left and right duals M^* and $*M$ are equal to $\text{Hom}(M, k)$ as a vector space, with the following H -action and H -coaction:*

(1) For $*M$,

$$\lambda_{*M}(\varphi) = \langle \varphi, m_{i(0)} \rangle s(m_{i(-1)}) \otimes m^i, \quad (3.4)$$

$$\begin{aligned}
h \cdot \varphi &= f(s^{-1}(h_3), m_{i(-1)})g((s^{-1}(h_2) \cdot m_{i(0)})_{(-1)}, s^{-1}(h_1)) \\
&\quad \varphi((s^{-1}(h_2) \cdot m_{i(0)})_{(0)})m^i.
\end{aligned} \tag{3.5}$$

(2) For M^* ,

$$\lambda_{M^*}(\varphi') = \langle \varphi', m_{i(0)} \rangle s^{-1}(m_{i(-1)}) \otimes m^i, \tag{3.6}$$

$$\begin{aligned}
h \cdot \varphi' &= f(s^{-1}(m_{i(-1)}), h_3)g(h_1, s^{-1}((s(h_2) \cdot m_{i(0)})_{(-1)})) \\
&\quad \varphi'((s(h_2) \cdot m_{i(0)})_{(0)})m^i,
\end{aligned} \tag{3.7}$$

for all $h \in H, \varphi \in {}^*M, \varphi' \in M^*$.

Proof. The left H -coaction on *M viewed as an object in ${}^H_H\mathcal{YD}$ is the same as the left H -coaction on *M viewed as an object in ${}^H\mathcal{M}$. Now we compute the left H -action using (3.1). For all $h \in H, \varphi \in {}^*M$,

$$\begin{aligned}
h \cdot \varphi &= (id \otimes \varepsilon)c_{H, {}^*M}(h \otimes \varphi) \\
&= \sigma^{-1}(m_{(-1)1}^i, m_{i(-1)2}, h_1 s((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)8}) \sigma(m_{i(-1)3}, h_2, s((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)7})) \\
&\quad \alpha(m_{i(-1)1}^i, q^L(s^{-1}(h_8), m_{i(-1)4}h_3) \sigma(s^{-1}(h_7), m_{i(-1)5}, h_4) p^R((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)1}, s^{-1}(h_5)) \\
&\quad \sigma^{-1}(h_9, (s^{-1}(h_6) \cdot m_{i(0)})_{(-1)2}, s((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)6})) \beta((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)4}) \\
&\quad \sigma(m_{(-1)2}^i, h_{10}, (s^{-1}(h_6) \cdot m_{i(0)})_{(-1)3} s((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)5})) \varphi((s^{-1}(h_6) \cdot m_{i(0)})_{(0)})m_{(0)}^i \\
&\stackrel{(1.5)(3.4)}{=} q^L(m_{i(-1)1}, h_1 s((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)4})) \sigma(m_{i(-1)2}, h_2, s((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)3})) \\
&\quad q^L(s^{-1}(h_8), m_{i(-1)3}h_3) \sigma(s^{-1}(h_7), m_{i(-1)4}, h_4) p^R((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)1}, s^{-1}(h_5)) \\
&\quad p^R(h_9, (s^{-1}(h_6) \cdot m_{i(0)})_{(-1)2}) \varphi((s^{-1}(h_6) \cdot m_{i(0)})_{(0)})m^i \\
&\stackrel{(3.2)}{=} q^L(m_{i(-1)1}, h_1 s((s^{-1}(h_5) \cdot m_{i(0)})_{(-1)4})) \sigma(m_{i(-1)2}, h_2, s((s^{-1}(h_5) \cdot m_{i(0)})_{(-1)3})) \\
&\quad q^L(s^{-1}(h_7), m_{i(-1)3}) \sigma^{-1}(h_8, s^{-1}(h_6) m_{i(-1)4}, h_3) p^R((s^{-1}(h_5) \cdot m_{i(0)})_{(-1)1}, s^{-1}(h_4)) \\
&\quad p^R(h_9, (s^{-1}(h_5) \cdot m_{i(0)})_{(-1)2}) \varphi((s^{-1}(h_5) \cdot m_{i(0)})_{(0)})m^i \\
&\stackrel{(2.10)}{=} q^L(m_{i(-1)1}, h_1 s((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)5})) \sigma(m_{i(-1)2}, h_2, s((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)4})) \\
&\quad q^L(s^{-1}(h_7), m_{i(-1)3}) \sigma^{-1}(h_8, (s^{-1}(h_6) \cdot m_{i(0)})_{(-1)1} s^{-1}(h_5), h_3) p^R((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)2}, s^{-1}(h_4)) \\
&\quad p^R(h_9, (s^{-1}(h_6) \cdot m_{i(0)})_{(-1)3}) \varphi((s^{-1}(h_6) \cdot m_{i(0)})_{(0)})m^i \\
&\stackrel{(3.2)}{=} q^L(m_{i(-1)1}, h_1 s((s^{-1}(h_5) \cdot m_{i(0)})_{(-1)5})) \sigma(m_{i(-1)2}, h_2, s((s^{-1}(h_5) \cdot m_{i(0)})_{(-1)4})) \\
&\quad q^L(s^{-1}(h_6), m_{i(-1)3}) \sigma(h_7, (s^{-1}(h_5) \cdot m_{i(0)})_{(-1)1}, s^{-1}(h_4)) p^R(h_8 (s^{-1}(h_5) \cdot m_{i(0)})_{(-1)2}, s^{-1}(h_3)) \\
&\quad p^R(h_9, (s^{-1}(h_5) \cdot m_{i(0)})_{(-1)3}) \varphi((s^{-1}(h_5) \cdot m_{i(0)})_{(0)})m^i \\
&\stackrel{(2.6)}{=} q^L(m_{i(-1)1}, h_1 s((s^{-1}(h_7) \cdot m_{i(0)})_{(-1)6})) \sigma(m_{i(-1)2}, h_2, s((s^{-1}(h_7) \cdot m_{i(0)})_{(-1)5})) \\
&\quad q^L(s^{-1}(h_8), m_{i(-1)3}) \sigma^{-1}(h_9((s^{-1}(h_7) \cdot m_{i(0)})_{(-1)1} s^{-1}(h_6)), h_3, s((s^{-1}(h_7) \cdot m_{i(0)})_{(-1)4}))
\end{aligned}$$

$$\begin{aligned}
& p^R(h_{10}, (s^{-1}(h_7) \cdot m_{i(0)})_{(-1)2} s^{-1}(h_5)) g((s^{-1}(h_7) \cdot m_{i(0)})_{(-1)3}, s^{-1}(h_4)) \varphi((s^{-1}(h_7) \cdot m_{i(0)})_{(0)}) m^i \\
& \stackrel{(2.10)}{=} q^L(m_{i(-1)1}, h_1 s((s^{-1}(h_5) \cdot m_{i(0)})_{(-1)4})) \sigma(m_{i(-1)2}, h_2, s((s^{-1}(h_5) \cdot m_{i(0)})_{(-1)3})) \\
& q^L(s^{-1}(h_8), m_{i(-1)3}) \sigma^{-1}(h_9(s^{-1}(h_7) m_{i(-1)4})), h_3, s((s^{-1}(h_5) \cdot m_{i(0)})_{(-1)2})) \\
& p^R(h_{10}, (s^{-1}(h_6) m_{i(-1)5})) g((s^{-1}(h_5) \cdot m_{i(0)})_{(-1)1}, s^{-1}(h_4)) \\
& (m^i \otimes h_{11}) \varphi((s^{-1}(h_5) \cdot m_{i(0)})_{(0)}) \\
& \stackrel{(2.2)}{=} q^L(m_{i(-1)1}, h_1 s((s^{-1}(h_5) \cdot m_{i(0)})_{(-1)4})) \sigma(m_{i(-1)2}, h_2, s((s^{-1}(h_5) \cdot m_{i(0)})_{(-1)3})) \\
& q^L(s^{-1}(h_7), m_{i(-1)4}) \sigma^{-1}(m_{i(-1)3}, h_3, s((s^{-1}(h_5) \cdot m_{i(0)})_{(-1)2})) \\
& p^R(h_8, (s^{-1}(h_6) m_{i(-1)5})) g((s^{-1}(h_5) \cdot m_{i(0)})_{(-1)1}, s^{-1}(h_4)) \varphi((s^{-1}(h_5) \cdot m_{i(0)})_{(0)}) m^i \\
& \stackrel{(1.10)}{=} q^L(m_{i(-1)1}, s(s^{-1}(h_3) m_{i(-1)4})) g((s^{-1}(h_2) \cdot m_{i(0)})_{(-1)}, s^{-1}(h_1)) \\
& q^L(s^{-1}(h_5), m_{i(-1)2}) p^R(h_6, s^{-1}(h_4) m_{i(-1)3}) \varphi((s^{-1}(h_2) \cdot m_{i(0)})_{(0)}) m^i \\
& \stackrel{(3.3)}{=} f(s^{-1}(h_3), m_{i(-1)}) (m^i \otimes h_4) g((s^{-1}(h_2) \cdot m_{i(0)})_{(-1)}, s^{-1}(h_1)) \varphi((s^{-1}(h_2) \cdot m_{i(0)})_{(0)}),
\end{aligned}$$

as claimed. The structure on M^* can be computed in a similar way. \square

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